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# EXISTENCE OF SUM AND PRODUCT INTEGRALS

BY

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This paper is dedicated to Professor H. S. Wall.

ABSTRACT. Functions are from $R \times R$ to $R$, where $R$ represents the set of real numbers. If $c$ is a number and either (1) $\int_a^b G^2$ exists and $\int_a^b G$ exists, (2) $\int_a^b G$ exists and ${}_a\Pi^b(1+G)$ exists and is not zero or (3) each of ${}_a\Pi^b(1+G)$ and ${}_a\Pi^b(1-G)$ exists and is not zero, then $\int_a^b cG$ exists, $\int_a^b |cG - \int cG| = 0$, ${}_x\Pi^y(1+cG)$ exists for $a \le x < y \le b$ and $\int_a^b |1 + cG - \Pi(1+cG)| = 0$. Furthermore, if $H$ is a function such that $\lim_{x\to p^-} H(x, p)$, $\lim_{x\to p^+} H(p, x)$, $\lim_{x,y\to p^-} H(x, y)$ and $\lim_{x,y\to p^+} H(x, y)$ exist for each $p \in [a, b]$, $n \ge 2$ is an integer, and $G$ satisfies either (1), (2) or (3) of the above, then $\int_a^b HG^n$ exists, $\int_a^b |HG^n - \int HG^n| = 0$, ${}_x\Pi^y(1+HG^n)$ exists for $a \le x < y \le b$ and $\int_a^b |1 + HG^n - \Pi(1+HG^n)| = 0$.

All integrals and definitions are of the subdivision-refinement type, and functions are from $R \times R$ to $R$, where $R$ represents the set of real numbers. Furthermore, functions are assumed to be defined only for elements $\{x, y\}$ of $R \times R$ such that $x \le y$, and $G(x, x) = 0$. The statements that $G$ is bounded, $G \in OP^\circ$, $G \in OQ^\circ$ and $G \in OB^\circ$ on $[a, b]$ mean that there exist a subdivision $D$ of $[a, b]$ and positive numbers $B$ and $c$ such that if $J = \{x_q\}_0^n$ is a refinement of $D$, then

(1) $|G(u)| < B$ for $u \in J(I)$,

(2) $|\Pi_i^j(1 + G_q)| < B$ for $1 \le i \le j \le n$,

(3) $|\Pi_i^j(1 + G_q)| > c$ for $1 \le i \le j \le n$, and

(4) $\sum_{J(I)} |G| < B$,

respectively, where $G_q = G(x_{q-1}, x_q)$ and $J(I) = \{[x_{q-1}, x_q]\}_1^n$. Similarly, statements of the form $G > b$ should be interpreted in terms of subdivisions and refinements. The symbols $G(p, p^+)$, $G(p^-, p)$, $G(p^+, p^+)$ and $G(p^-, p^-)$ are used to denote $\lim_{x\to p^+} G(p, x)$, $\lim_{x\to p^-} G(x, p)$, $\lim_{x,y\to p^+} G(x, y)$, and $\lim_{x,y\to p^-} G(x, y)$, respectively. Further, $G \in OL^\circ$ on $[a, b]$ only if $G(p, p^+)$, $G(p^-, p)$, $G(p^+, p^+)$ and $G(p^-, p^-)$ exist for each $p \in [a, b]$, and $G \in OL^{14}$ on $[a, b]$ only if $G \in OL^\circ$ on $[a, b]$ and $G(p^+, p^+) = G(p^-, p^-) = 0$ for each $p \in [a, b]$. For convenience in notation, when we consider a function $G$ defined only

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on intervals $[x, y]$ such that $a \le x < y \le b$, we adopt the convention that

$$G(a-, a-) = G(a-, a) = G(b^+, b^+) = G(b, b^+) = 0.$$

Also, $G \in OA^\circ$ on $[a, b]$ only if $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$, and $G \in OM^\circ$ on $[a, b]$ only if ${}_x\Pi^y (1 + G)$ exists for $a \le x < y \le b$ and $\int_a^b |1 + G - \Pi(1 + G)| = 0$. The sources of these definitions are [3, p. 299], [4, p. 493], [5] and [7].

**Lemma 1.1.** *If $\int_a^b G^2$ exists and $\int_a^b G$ exists, then $G \in OL^{14}$ on $[a, b]$.*

The proof of Lemma 1.1 is straightforward, and therefore we omit it.

**Lemma 1.2.** *If $\beta > 0$, $|G| < 1 - \beta$ on $[a, b]$, $\int_a^b G^2$ exists and $\int_a^b G$ exists, then ${}_a\Pi^b (1 + G)$ exists and is not zero* [6, *Theorem* 5].

**Lemma 1.3.** *If $G$ is bounded on $[a, b]$ and ${}_a\Pi^b (1 + G)$ exists and is not zero, then $G \in OP^\circ$ and $OQ^\circ$ on $[a, b]$* [7, *Theorem* 2].

**Lemma 1.4.** *If $G$ is bounded on $[a, b]$ and ${}_a\Pi^b (1 + G)$ exists and is not zero, then $G \in OM^\circ$ on $[a, b]$.*

**Indication of Proof.** This lemma follows from Lemma 1.3 and a result of B. W. Helton [3, Theorem 4.2, p. 305].

**Lemma 1.5.** *If $\int_a^b G$ exists, then $G \in OA^\circ$ on $[a, b]$.*

This result is due to A. Kolmogoroff [8, p. 669]. The reader is also referred to results by W. D. L. Appling [1, Theorems 1, 2, p. 155] and B. W. Helton [3, Theorem 4.1, p. 304].

**Lemma 1.6.** *If $E = \{p_i\}_1^r$ is a set of distinct points from $[a, b]$ and $F$, $G$ and $H$ are functions defined on $[a, b]$ such that*

(1) *$G \in OL^\circ$ on $[a, b]$ and $\int_a^b G$ exists,*

(2) *if $p \in E$, then $H(p, p^+)$ and $H(p^-, p)$ exist, and*

(3) *if $a \le x < y \le b$, then $F(x, y) = G(x, y)$ if $x \notin E$ and $y \notin E$ and $F(x, y) = H(x, y)$ if $x \in E$ or $y \in E$,*

*then $\int_a^b F$ exists and is*

$$\int_a^b G + \sum_{p \in E} [H(p, p^+) + H(p^-, p) - G(p, p^+) - G(p^-, p)].$$

The proof of Lemma 1.6 is straightforward, and therefore we omit it.

**Lemma 1.7.** *If* $E = \{p_i\}_1^r$ *is a set of distinct points from* $[a, b]$ *and* $F$, $G$ *and* $H$ *are functions on* $[a, b]$ *such that*

(1) $G \in OL^\circ$ *on* $[a, b]$ *and* ${}_a\Pi^b\,(1 + G)$ *exists and is not zero,*

(2) *if* $p \in E$, *then* $H(p, p^+)$ *and* $H(p^-, p)$ *exist, and*

(3) *if* $a \le x < y \le b$, *then* $F(x, y) = G(x, y)$ *if* $x \notin E$ *and* $y \notin E$ *and* $F(x, y) = H(x, y)$ *if* $x \in E$ *or* $y \in E$,

*then* ${}_a\Pi^b\,(1 + F)$ *exists and is*

$$\left\{{}_a\prod^b (1 + G)\right\} \cdot \left\{\prod_{p\in E} [1 + H(p, p^+)][1 + H(p^-, p)]\right\} \cdot \left\{\prod_{p\in E} [1 + G(p, p^+)][1 + G(p^-, p)]\right\}^{-1}.$$

*Furthermore,* $F \in OM^\circ$ *on* $[a, b]$.

**Proof.** We first show that ${}_a\Pi^b\,(1 + F)$ exists and is $P$, where

$$P = \left[{}_a\prod^b (1 + G)\right][P_1][P_2]^{-1},$$

$$P_1 = \prod_{p\in E} [1 + H(p, p^+)][1 + H(p^-, p)]$$

and

$$P_2 = \prod_{p\in E} [1 + G(p, p^+)][1 + G(p^-, p)].$$

Since $G \in OL^\circ$ on $[a, b]$, it follows from the covering theorem that $G$ is bounded on $[a, b]$. Hence, it follows from Lemma 1.3 that $G \in OP^\circ$ and $OQ^\circ$ on $[a, b]$. Let $\epsilon > 0$. There exist a subdivision $D_1$ of $[a, b]$ and a number $B > 1$ such that if $J = \{x_q\}_0^n$ is a refinement of $D_1$, then

(1) $|\Pi_i^j\,(1 + G_q)| < B$ for $1 \le i \le j \le n$,

(2) $|P_1| < B$ and $|[P_2]^{-1}| < B$, and

(3) $|{}_a\Pi^b\,(1 + G) - \Pi_{J(I)}\,(1 + G)| < \epsilon(3B^2)^{-1}$.

Let $\delta$ be a positive number such that if

$$p_i \overset{a}{-} \delta\} < x_i < p_i < y_i < \{p_i \overset{b}{+} \delta$$

for $1 \le i \le r$, then

$$\left|1 - \{P_2\}^{-1}\left\{\prod_{i=1}^{r} [1 + G(x_i, p_i)][1 + G(p_i, y_i)]\right\}\right| < \epsilon(3B^{2r+1})^{-1}$$

and

$$\left|P_1 - \prod_{i=1}^{r} [1 + H(x_i, p_i)][1 + H(p_i, y_i)]\right| < \epsilon(3B^{2r})^{-1}.$$

Let $D$ denote the subdivision of $[a, b]$ consisting of

$$D_1 \cup \{p_i - \delta\}_1^r \cup \{p_i + \delta\}_1^r \cup \{\tfrac{1}{2}(p_i + p_{i+1})\}_1^{r-1} \cup E$$

less any elements which are not in $[a, b]$. Suppose $J$ is a refinement of $D$. Let $K(I)$ be the subset of $J(I)$ such that $u \in K(I)$ only if neither end point of $u$ belongs to $E$. Note that no interval in $J(I)$ can have elements of $E$ at both end points. Let $L(I) = J(I) - K(I)$. Thus,

$$\begin{aligned}
\left|\prod_{J(I)} (1 + F) - P\right| &\le \left|\prod_{J(I)} (1 + F) - \left[\prod_{J(I)} (1 + G)\right][P_1][P_2]^{-1}\right| \\
&\quad + \left|\left[\prod_{J(I)} (1 + G)\right][P_1][P_2]^{-1} - P\right| \\
&< \left|\prod_{K(I)} (1 + G)\right| \left|\prod_{L(I)} (1 + F) - \left[\prod_{L(I)} (1 + G)\right][P_1][P_2]^{-1}\right| + \epsilon(3B^2)^{-1}B^2 \\
&\le B^{2r}\left|\prod_{L(I)} (1 + F) - \left[\prod_{L(I)} (1 + G)\right][P_1][P_2]^{-1}\right| + \frac{\epsilon}{3} \\
&\le B^{2r}\left|\prod_{L(I)} (1 + F) - P_1\right| + B^{2r}|P_1|\left|1 - \left[\prod_{L(I)} (1 + G)\right][P_2]^{-1}\right| + \frac{\epsilon}{3} \\
&< \epsilon(3B^{2r})^{-1}B^{2r} + \epsilon(3B^{2r+1})^{-1}B^{2r+1} + \epsilon/3 = \epsilon.
\end{aligned}$$

Therefore, ${}_a\Pi^b (1 + F)$ exists and is $P$.

We now show that $F \in OM^\circ$ on $[a, b]$. Since it can be shown that ${}_x\Pi^y (1 + F)$ exists for $a \le x < y \le b$ by an argument similar to the one used in the preceding paragraph, it is only necessary to show that

$$\int_a^b \left|1 + F - \prod (1 + F)\right| = 0.$$

Let $\epsilon > 0$. As noted in the previous paragraph, $G \in OP^\circ$ and $OQ^\circ$ on $[a, b]$. Hence, Lemma 1.4 implies that $G \in OM^\circ$ on $[a, b]$. There exist a subdivision $D_1$ of $[a, b]$ and a number $B > 1$ such that if $\{x_q\}_0^n$ is a refinement of $D_1$, then

(1) $|\Pi_i^j (1 + G_q)| < B$ for $1 \leq i \leq j \leq n$,

(2) $|1 + F_q| < B$ and $|1 + G_q| > 1/B$ for $1 \leq q \leq n$, and

(3) $\Sigma_{q=1}^n |1 + G_q - \Pi_{J_q(I)} (1 + G)| < \epsilon(5B^2)^{-1}$, where $J_q$ is a subdivision of $[x_{q-1}, x_q]$ for $1 \leq q \leq n$.

Let $\delta$ be a positive number such that if $1 \leq i \leq r$ and

$$p_i \overset{a}{-} \delta\} < x_i' < x_i < p_i < y_i' < y_i < \{p_i \overset{b}{+} \delta,$$

then

(1) $|F(x_i, p_i) - F(x_i', p_i)| < \epsilon(10r)^{-1}$,

(2) $|F(p_i, y_i) - F(p_i, y_i')| < \epsilon(10r)^{-1}$,

(3) $|G(x_i, p_i) - G(x_i', p_i)| < \epsilon(10rB^3)^{-1}$, and

(4) $|G(p_i, y_i) - G(p_i, y_i')| < \epsilon(10rB^3)^{-1}$.

Let $D$ denote the subdivision of $[a, b]$ consisting of

$$D_1 \cup \{p_i - \delta\}_1^r \cup \{p_i + \delta\}_1^r \cup \{\tfrac{1}{2}(p_i + p_{i+1})\}_1^{r-1} \cup E$$

less any elements which are not in $[a, b]$.

Suppose $J = \{x_q\}_0^n$ is a refinement of $D$. For $1 \leq q \leq n$, let $J_q = \{x_{q,i}\}_0^{n(q)}$ be a subdivision of $[x_{q-1}, x_q]$ such that

$$\left| {}_{x_{q-1}}\prod^{x_q} (1 + F) - \prod_{J_q(I)} (1 + F) \right| < \frac{\epsilon}{5n}.$$

Also, for $1 \leq q \leq n$, suppose

(1) $q \in U$ only if $[x_{q-1}, x_q]$ does not have a point of $E$ as an end point,

(2) $q \in V(1)$ only if $x_q \in E$, and

(3) $q \in V(2)$ only if $x_{q-1} \in E$.

Note that $D$ is defined so that $q$ cannot belong to both $V(1)$ and $V(2)$. For $q \in V(1)$, let

(1) $K_q = \{x_{q,i}\}_0^{n(q)-1}$,

(2) $F_q' = F(x_{q,n(q)-1}, x_{q,n(q)})$, and

(3) $G_q' = G(x_{q,n(q)-1}, x_{q,n(q)})$,

and for $q \in V(2)$, let

(1) $K_q = \{x_{q,i}\}_1^{n(q)}$,

(2) $F_q' = F(x_{q,0}, x_{q,1})$, and

(3) $G_q' = G(x_{q,0}, x_{q,1})$.

If $q \in V(1)$ or $V(2)$, then $\Pi_{J_q(I)} (1 + F) = (1 + F_q')\Pi_{K_q(I)} (1 + G)$ and $\Pi_{J_q(I)} (1 + G) = (1 + G_q')\Pi_{K_q(I)} (1 + G)$. Thus,

$$\sum_{q=1}^{n}\left|1+F_q-{}_{x_{q-1}}\prod^{x_q}(1+F)\right| < \sum_{q=1}^{n}\left|1+F_q-\prod_{J_q(I)}(1+F)\right|+\frac{\epsilon}{5}$$

$$= \sum_{q\epsilon U}\left|1+G_q-\prod_{J_q(I)}(1+G)\right| + \sum_{i=1}^{2}\sum_{q\epsilon V(i)}\left|1+F_q-\prod_{J_q(I)}(1+F)\right|+\frac{\epsilon}{5}$$

$$< \frac{\epsilon}{5}+\sum_{i=1}^{2}\sum_{q\epsilon V(i)}\left|1+F_q-(1+F_q')\prod_{K_q(I)}(1+G)\right|+\frac{\epsilon}{5}$$

$$\leq \sum_{i=1}^{2}\sum_{q\epsilon V(i)}|1+F_q'|\left|1-\prod_{K_q(I)}(1+G)\right|+\sum_{i=1}^{2}\sum_{q\epsilon V(i)}|F_q-F_q'|+\frac{2\epsilon}{5}$$

$$< B\sum_{i=1}^{2}\sum_{q\epsilon V(i)}\left|1-\prod_{K_q(I)}(1+G)\right|+[2r][\epsilon(10r)^{-1}]+\frac{2\epsilon}{5}$$

$$= B\sum_{i=1}^{2}\sum_{q\epsilon V(i)}|(1+G_q)^{-1}|\left|1+G_q-(1+G_q)\prod_{K_q(I)}(1+G)\right|+\frac{3\epsilon}{5}$$

$$\leq B^2\sum_{i=1}^{2}\sum_{q\epsilon V(i)}\left|1+G_q-(1+G_q)\prod_{K_q(I)}(1+G)\right|+\frac{3\epsilon}{5}$$

$$\leq B^2\sum_{i=1}^{2}\sum_{q\epsilon V(i)}\left|1+G_q-\prod_{J_q(I)}(1+G)\right|$$

$$+ B^2\sum_{i=1}^{2}\sum_{q\epsilon V(i)}|G_q-G_q'|\left|\prod_{K_q(I)}(1+G)\right|+\frac{3\epsilon}{5}$$

$$< B^2[\epsilon(5B^2)^{-1}]+[2rB^3][\epsilon(10rB^3)^{-1}]+\frac{3\epsilon}{5}=\epsilon.$$

Therefore, $F \in \dot{OM}^\circ$ on $[a, b]$.

**Theorem 1.** *If $\int_a^b G^2$ exists, $\int_a^b G$ exists and $c$ is a number, then $cG \in OM^\circ$ and $OA^\circ$ on $[a, b]$.*

**Proof.** Since it follows from Lemma 1.5 that $cG \in OA^\circ$ on $[a, b]$, we need only show that $cG \in OM^\circ$ on $[a, b]$. Since $G \in OL^{14}$ on $[a, b]$ [Lemma 1.1], there exists a subdivision $D = \{x_q\}_0^n$ of $[a, b]$ such that if $1 \leq q \leq n$ and $x_{q-1} < x < y < x_q$, then $|cG(x, y)| < \frac{1}{2}$. Let $F$ be the function such that

(1) $F(x, y) = cG(x, y)$ if $x \notin D$ and $y \notin D$, and

(2) $F(x, y) = 0$ if $x \in D$ or $y \in D$.

Thus, $|F| < \frac{1}{2}$, and Lemma 1.6 implies that $\int_a^b F^2$ exists and $\int_a^b F$ exists. Therefore, ${}_a\Pi^b\,(1 + F)$ exists and is not zero by Lemma 1.2, and hence, Lemma 1.7 implies that $cG \in OM^\circ$ on $[a, b]$.

**Lemma 2.1.** *If* $G \in OM^\circ$ *or* $OA^\circ$ *on* $[a, b]$, $G \in OB^\circ$ *on* $[a, b]$ *and* $H \in OL^\circ$ *on* $[a, b]$, *then* $HG \in OM^\circ$ *and* $OA^\circ$ *on* $[a, b]$ [4, *Theorem* 2, *p.* 494], [3, *Theorem* 3.4, *p.* 301].

**Theorem 2.** *If* $\int_a^b G^2$ *exists,* $\int_a^b G$ *exists,* $H \in OL^\circ$ *on* $[a, b]$ *and* $n \geq 2$ *is an integer, then* $HG^n \in OM^\circ$ *and* $OA^\circ$ *on* $[a, b]$.

**Proof.** Since $G \in OL^{14}$ on $[a, b]$ [Lemma 1.1], it follows that $HG^{n-2} \in OL^\circ$ on $[a, b]$. Therefore, since $G^2 \in OA^\circ$ on $[a, b]$ [Lemma 1.5] and $G^2 \in OB^\circ$ on $[a, b]$, it follows from Lemma 2.1 that $HG^n \in OM^\circ$ and $OA^\circ$ on $[a, b]$.

Theorem 2 is not true for $n = 1$. The author [5, Theorem 10] has shown that for $\int_a^b HG$ to exist for every $H \in OL^\circ$ on $[a, b]$ it is necessary that $G \in OB^\circ$ on $[a, b]$. However, Davis and Chatfield [2, p. 747] define a function $G$ such that $\int_a^b G^2$ exists, $\int_a^b G$ exists and $G \notin OB^\circ$ on $[a, b]$.

**Lemma 3.1.** *If* ${}_a\Pi^b\,(1 + G)$ *exists and is not zero and* $\int_a^b G$ *exists, then* $G \in OL^{14}$ *on* $[a, b]$ [6, *Theorem* 9].

The conclusion of Theorem 9 [6] states that $G \in OL^\circ$ on $[a, b]$. However, the argument used to establish this also establishes that $G \in OL^{14}$ on $[a, b]$.

**Lemma 3.2.** *If* $\int_a^b HG$ *exists,* $G \geq 0$ *on* $[a, b]$, $H \in OL^\circ$ *on* $[a, b]$ *and* $H$ *is bounded away from zero on* $[a, b]$, *then* $\int_a^b G$ *exists.*

**Proof.** There exists a subdivision $\{x_q\}_0^n$ of $[a, b]$ such that if $1 \leq q \leq n$, then $H$ does not change sign on $(x_{q-1}, x_q)$. Hence, $HG \in OB^\circ$ on $[x_{q-1}, x_q]$, and thus, $HG \in OB^\circ$ on $[a, b]$. Therefore, since $H^{-1} \in OL^\circ$ on $[a, b]$ and $HG \in OA^\circ$ on $[a, b]$ [Lemma 1.5], it follows from Lemma 2.1 that $\int_a^b G$ exists.

**Lemma 3.3.** *If* ${}_a\Pi^b\,(1 + G)$ *exists and is not zero and* $G > -1$ *on* $[a, b]$, *then* $\int_a^b \ln(1 + G)$ *exists* [6, *Theorem* 4].

**Lemma 3.4.** *If* ${}_a\Pi^b\,(1 + G)$ *exists and is not zero and* $\int_a^b G$ *exists, then* $\int_a^b G^2$ *exists.*

**Proof.** Since $G \in OL^{14}$ on $[a, b]$ [Lemma 3.1], there exists a subdivision $D = \{x_q\}_0^n$ of $[a, b]$ such that if $1 \leq q \leq n$ and $x_{q-1} < x < y < x_q$, then $|G(x, y)| < ½$. Let $F$ be the function such that

(1) $F(x, y) = G(x, y)$ if $x \notin D$ and $y \notin D$, and

(2) $F(x, y) = 0$ if $x \in D$ or $y \in D$.

Therefore, ${}_a\Pi^b\,(1 + F)$ exists and is not zero by Lemma 1.7, and $\int_a^b F$ exists by Lemma 1.6. Thus, from Lemma 3.3, $\int_a^b \ln(1 + F) = \int_a^b \sum_{n=1}^\infty (-1)^{n-1} F^n/n$ exists, and hence,

$$\int_a^b \sum_{n=2}^{\infty} (-1)^{n-1} \frac{F^n}{n} = \int_a^b F^2 \left[ -\frac{1}{2} + \sum_{n=3}^{\infty} (-1)^{n-1} \frac{F^{n-2}}{n} \right]$$

exists. Thus, since $-\frac{1}{2} + \sum_{n=3}^{\infty} (-1)^{n-1} F^{n-2}/n$ is in $OL^{\circ}$ on $[a, b]$ and is bounded away from zero on $[a, b]$, Lemma 3.2 implies that $\int_a^b F^2$ exists. Therefore, $\int_a^b G^2$ exists by Lemma 1.6.

**Theorem 3.** *If* ${}_a\Pi^b (1 + G)$ *exists and is not zero,* $\int_a^b G$ *exists and* $c$ *is a number, then* $cG \in OM^{\circ}$ *and* $OA^{\circ}$ *on* $[a, b]$.

**Proof.** It follows from Lemma 3.4 that $\int_a^b G^2$ exists. Therefore, Theorem 3 follows from Theorem 1.

**Theorem 4.** *If* ${}_a\Pi^b (1 + G)$ *exists and is not zero,* $\int_a^b G$ *exists,* $H \in OL^{\circ}$ *on* $[a, b]$ *and* $n \geq 2$ *is an integer, then* $HG^n \in OM^{\circ}$ *and* $OA^{\circ}$ *on* $[a, b]$.

**Proof.** It follows from Lemma 3.4 that $\int_a^b G^2$ exists. Therefore, Theorem 4 follows from Theorem 2.

**Lemma 5.1.** *If each of* ${}_a\Pi^b (1 + G)$ *and* ${}_a\Pi^b (1 - G)$ *exists and is not zero, then* $G$ *is bounded on* $[a, b]$ [7, *Lemma* 6.1].

**Lemma 5.2.** *If each of* ${}_a\Pi^b (1 + G)$ *and* ${}_a\Pi^b (1 - G)$ *exists and is not zero, then* $G \in OL^{14}$ *on* $[a, b]$.

**Proof.** Let $p \in (a, b]$. It follows from Lemmas 5.1 and 1.3 that $G \in OQ^{\circ}$ and $-G \in OQ^{\circ}$ on $[a, b]$. By applying this result and the Cauchy criterion for product integrals, we have that

(1) $0 = \lim_{x,y\to p^-} \{[1 + G(x, p)] - [1 + G(x, y)][1 + G(y, p)]\}$
$= \lim_{x,y\to p^-} \{G(x, p) - G(x, y) - G(y, p) - G(x, y)G(y, p)\}$, and

(2) $0 = \lim_{x,y\to p^-} \{[1 - G(x, p)] - [1 - G(x, y)][1 - G(y, p)]\}$
$= \lim_{x,y\to p^-} \{-G(x, p) + G(x, y) + G(y, p) - G(x, y)G(y, p)\}$.

Thus,

(3) $0 = \lim_{x,y\to p^-} [G(x, p) - G(x, y) - G(y, p)]$, and

(4) $0 = \lim_{x,y\to p^-} G(x, y)G(y, p)$.

Note that limits (1), (3) and (4) are the same as limits (2), (1) and (3), respectively, in Theorem 9 of a previous paper of the author [6]. It follows that $G(p^-, p)$ exists by the argument used in Theorem 9 [6]. Thus, it follows from (3) and the existence of $G(p^-, p)$ that $G(p^-, p^-)$ exists and is zero. Therefore, since right limits can be treated similarly, $G \in OL^{14}$ on $[a, b]$.

**Lemma 5.3.** *If each of* ${}_a\Pi^b\,(1 + G)$ *and* ${}_a\Pi^b\,(1 - G)$ *exists and is not zero, then* $\int_a^b G^2$ *exists.*

**Proof.** Since each of ${}_a\Pi^b\,(1 + G)$ and ${}_a\Pi^b\,(1 - G)$ exists and is not zero, ${}_a\Pi^b\,(1 - G^2)$ exists and is not zero. Thus, since $G$ is bounded on $[a, b]$ [Lemma 5.1], $-G^2 \in OQ^\circ$ on $[a, b]$ [Lemma 1.3]. Since $G \in OL^{14}$ on $[a, b]$ [Lemma 5.2], there exists a subdivision $\{x_q\}_0^n$ of $[a, b]$ such that if $1 \leq q \leq n$ and $x_{q-1} < x < y < x_q$, then $|G(x, y)| < 1$. Thus, since $-G^2 \in OQ^\circ$ on $[a, b]$, it follows that $G^2 \in OB^\circ$ on $[x_{q-1}, x_q]$ for $1 \leq q \leq n$. This is true because if $F$ is a function such that $0 \leq F \leq 1$ on an interval $[r, s]$ and $F \notin OB^\circ$ on $[r, s]$, then $-F \notin OQ^\circ$ on $[r, s]$. Therefore, $G^2 \in OB^\circ$ on $[a, b]$. Lemma 1.4 implies that $-G^2 \in OM^\circ$ on $[a, b]$. Hence, it follows from Lemma 2.1 that $\int_a^b G^2$ exists.

**Lemma 5.4.** *If* $\beta > 0$, $|G| < 1 - \beta$ *on* $[a, b]$, $\int_a^b G^2$ *exists and* ${}_a\Pi^b\,(1 + G)$ *exists and is not zero, then* $\int_a^b G$ *exists* [6, *Theorem* 5].

**Theorem 5.** *If each of* ${}_a\Pi^b\,(1 + G)$ *and* ${}_a\Pi^b\,(1 - G)$ *exists and is not zero and* $c$ *is a number, then* $cG \in OM^\circ$ *and* $OA^\circ$ *on* $[a, b]$.

**Proof.** Since $G \in OL^{14}$ on $[a, b]$ [Lemma 5.2], there exists a subdivision $D = \{x_q\}_0^n$ of $[a, b]$ such that if $1 \leq q \leq n$ and $x_{q-1} < x < y < x_q$, then $|G(x, y)| < \frac{1}{2}$. Let $F$ be the function such that

(1) $F(x, y) = G(x, y)$ if $x \notin D$ and $y \notin D$, and

(2) $F(x, y) = 0$ if $x \in D$ or $y \in D$.

Since $\int_a^b G^2$ exists [Lemma 5.3], $\int_a^b F^2$ also exists [Lemma 1.6]. Further, since $G \in OL^{14}$ on $[a, b]$ and ${}_a\Pi^b\,(1 + G)$ exists and is not zero, ${}_a\Pi^b\,(1 + F)$ exists and is not zero [Lemma 1.7]. Therefore, $\int_a^b F$ exists [Lemma 5.4], and hence, $\int_a^b G$ exists [Lemma 1.6]. Thus, since $\int_a^b G^2$ exists and $\int_a^b G$ exists, it follows from Theorem 1 that $cG \in OM^\circ$ and $OA^\circ$ on $[a, b]$.

**Theorem 6.** *If each of* ${}_a\Pi^b\,(1 + G)$ *and* ${}_a\Pi^b\,(1 - G)$ *exists and is not zero,* $H \in OL^\circ$ *on* $[a, b]$ *and* $n \geq 2$ *is an integer, then* $HG^n \in OM^\circ$ *and* $OA^\circ$ *on* $[a, b]$.

**Proof.** It follows as in Theorem 5 that $\int_a^b G^2$ exists and $\int_a^b G$ exists. Hence, Theorem 2 implies that $HG^n \in OM^\circ$ and $OA^\circ$ on $[a, b]$.

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